Lesson 12. Dynamic Programming – Review

- Recall from Lessons 5-11:
  - A **dynamic program** models situations where decisions are made in a sequential process in order to optimize some objective
  - **Stages** $t = 1, 2, \dots, T$
    - stage $T \leftrightarrow$ end of decision process
  - **States** $n = 0, 1, \dots, N \leftarrow$ possible conditions of the system at each stage
  - Two representations: **shortest/longest path** and **recursive**

Shortest/longest path		Recursive
node t_n	$\leftrightarrow$	state n at stage t
edge $(t_n, (t+1)_m)$	$\leftrightarrow$	allowable decision x_t in state n at stage t that results in being in state m at stage $t+1$
length of edge $(t_n, (t+1)_m)$	$\leftrightarrow$	contribution of decision x_t in state n at stage t that results in being in state m at stage $t+1$
length of shortest/longest path from node t_n to end node	$\leftrightarrow$	value-to-go function $f_t(n)$
length of edges (T_n, end)	$\leftrightarrow$	boundary conditions $f_T(n)$
shortest or longest path	$\leftrightarrow$	recursion is min or max: $f_t(n) = \min\limits_{x_t \text{ allowable}} \text{ or } \max \left\{ \begin{pmatrix} \text{contribution of} \\ \text{decision } x_t \end{pmatrix} + f_{t+1} \begin{pmatrix} \text{new state} \\ \text{resulting} \\ \text{from } x_t \end{pmatrix} \right\}$
source node 1_n	$\leftrightarrow$	desired value-to-go function value $f_1(n)$

Example 1. Simplexville Oil needs to build capacity to refine 1,000 barrels of oil and 2,000 barrels of gasoline per day. Simplexville can build a refinery at 2 locations. The cost of building a refinery is as follows:

Oil capacity per day	Gas capacity per day	Building cost ($ millions)
0	0	0
1000	0	5
0	1000	7
1000	1000	14

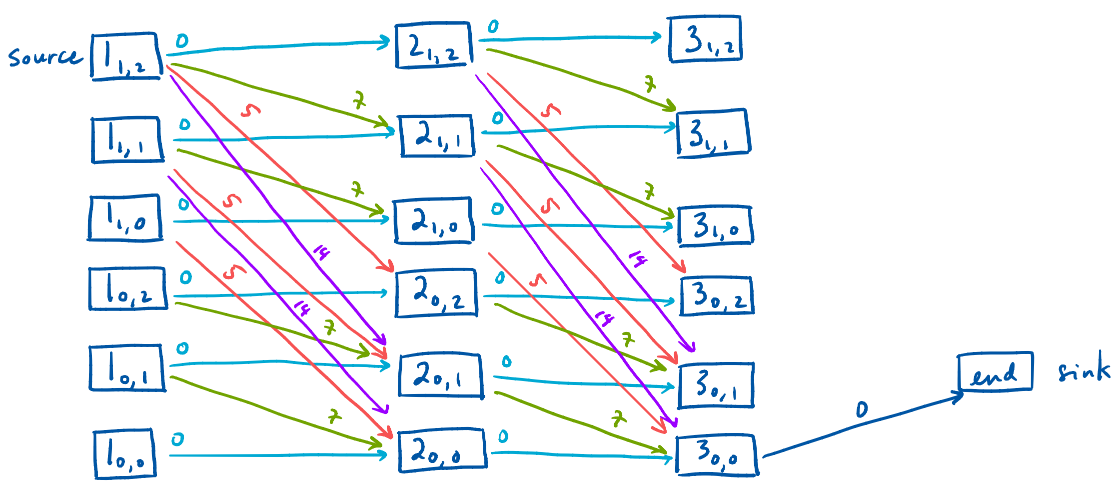
The problem is to determine how much capacity should be built at each location in order to minimize the total building cost. Assume that the capacity requirements must be met exactly.

a. Formulate this problem as a dynamic program by giving its shortest path representation.
b. Formulate this problem as a dynamic program by giving its recursive representation. Solve the dynamic program.

Stage t $\leftrightarrow$ $\begin{cases} \text{deciding to build at location } t & t = 1, 2 \\ \text{end of decision-making process} & t = 3 \end{cases}$

State (n_1, n_2) $\leftrightarrow$ n_1 oil capacity and n_2 gas capacity still needed to be built $n_1 = 0, 1$ $n_2 = 0, 1, 2$

Find shortest path:



Recursive representation

- Stage t $\longleftrightarrow$ $\begin{cases} \text{deciding to build at location } t & t=1,2 \\ \text{end of decision-making process} & t=3 \end{cases}$

- State (n_1, n_2) $\longleftrightarrow$ n_1 oil capacity and n_2 gas capacity still needed to be built $\quad n_1 = 0,1$, $n_2 = 0,1,2$

- Allowable decisions x_t at stage t and state (n_1, n_2):

  $x_t = (x_{t1}, x_{t2})$ $\longleftrightarrow$ build x_{t1} oil capacity and x_{t2} gas capacity at location t

  x_t must satisfy:
  $$x_{t1} \in \{0,1\}$$
  $$x_{t2} \in \{0,1\}$$
  $$\left.\begin{array}{l} x_{t1} \le n_1 \\ x_{t2} \le n_2 \end{array}\right\} \text{can't overbuild capacity}$$
  for $t = 1,2$, $n_1 = 0,1$, $n_2 = 0,1,2$

- Contribution of x_t at stage t and state (n_1, n_2):

  $$c(x_{t1}, x_{t2}) = \begin{cases} 0 & \text{if } (x_{t1}, x_{t2}) = (0,0) \\ 5 & \text{if } (x_{t1}, x_{t2}) = (1,0) \\ 7 & \text{if } (x_{t1}, x_{t2}) = (0,1) \\ 14 & \text{if } (x_{t1}, x_{t2}) = (1,1) \end{cases}$$
  for $t = 1,2$, $n_1 = 0,1$, $n_2 = 0,1,2$

- Value-to-go function

  $f_t(n_1, n_2)$ = minimum total cost to build n_1 oil capacity and n_2 gas capacity with locations $t, \ldots, 2$ available
  for $t = 1,2,3$; $n_1 = 0,1$; $n_2 = 0,1,2$

- Boundary conditions: $f_3(n_1, n_2) = \begin{cases} 0 & \text{if } (n_1, n_2) = (0,0) \\ +\infty & \text{o/w} \end{cases}$ for $n_1 = 0,1$; $n_2 = 0,1,2$.

- Recursion:

  $$f_t(n_1, n_2) = \min_{\substack{x_{t1} \in \{0,1\} \\ x_{t2} \in \{0,1\} \\ x_{t1} \le n_1 \\ x_{t2} \le n_2}} \left\{ c(x_{t1}, x_{t2}) + f_{t+1}(n_1 - x_{t1}, n_2 - x_{t2}) \right\}$$
  for $t = 1,2$, $n_1 = 0,1$, $n_2 = 0,1,2$

  oil, gas
  state n_1, n_2
  decision x_{t1}, x_{t2}
  new state $n_1 - x_{t1}, n_2 - x_{t2}$

- Desired value-to-go function value: $f_1(1,2)$

$$f_t(n_1,n_2) = \min_{\substack{x_{t1}\in\{0,1\}\\ x_{t2}\in\{0,1\}\\ x_{t1}\le n_1\\ x_{t2}\le n_2}} \left\{ c(x_{t1},x_{t2}) + f_{t+1}(n_1-x_{t1},\, n_2-x_{t2}) \right\} \qquad c(x_{t1},x_{t2}) = \begin{cases} 0 & \text{if } (x_{t1},x_{t2})=(0,0) \\ 5 & \text{if } (x_{t1},x_{t2})=(1,0) \\ 7 & \text{if } (x_{t1},x_{t2})=(0,1) \\ 14 & \text{if } (x_{t1},x_{t2})=(1,1) \end{cases} \quad \text{for } \begin{array}{l} t=1,2 \\ n_1=0,1 \\ n_2=0,1,2 \end{array}$$

Solving backwards

Stage 3:
(boundary conditions)

$$f_3(n_1,n_2) = \begin{cases} 0 & \text{if } (n_1,n_2)=(0,0) \\ +\infty & \text{o/w} \end{cases} \qquad \text{for } \begin{array}{l} n_1=0,1 \\ n_2=0,1,2 \end{array}$$

Stage 2:

$$f_2(1,2) = \min_{\substack{x_{21}\in\{0,1\}\\ x_{22}\in\{0,1\}\\ x_{21}\le 1\\ x_{22}\le 2}} \left\{ c(x_{21},x_{22}) + f_3(1-x_{21},\, 2-x_{22}) \right\} = \min\left\{ \begin{array}{ll} \underset{(0,0)}{\overset{0+\infty}{c(0,0)+f_3(1,2)}}, & \underset{(1,0)}{\overset{5+\infty}{c(1,0)+f_3(0,2)}} \\ \underset{(0,1)}{\overset{7+\infty}{c(0,1)+f_3(1,1)}}, & \underset{(1,1)}{\overset{14+\infty}{c(1,1)+f_3(0,1)}} \end{array} \right\} = +\infty$$

$$f_2(1,1) = \min_{\substack{x_{21}\in\{0,1\}\\ x_{22}\in\{0,1\}\\ x_{21}\le 1\\ x_{22}\le 1}} \left\{ c(x_{21},x_{22}) + f_3(1-x_{21},\, 1-x_{22}) \right\} = \min\left\{ \begin{array}{ll} \underset{(0,0)}{\overset{0+\infty}{c(0,0)+f_3(1,1)}}, & \underset{(1,0)}{\overset{5+\infty}{c(1,0)+f_3(0,1)}} \\ \underset{(0,1)}{\overset{7+\infty}{c(0,1)+f_3(1,0)}}, & \underset{(1,1)}{\overset{14+0}{c(1,1)+f_3(0,0)}} \end{array} \right\} = 14$$

$$f_2(1,0) = \min_{\substack{x_{21}\in\{0,1\}\\ x_{22}\in\{0,1\}\\ x_{21}\le 1\\ x_{22}\le 0}} \left\{ c(x_{21},x_{22}) + f_3(1-x_{21},\, 0-x_{22}) \right\} = \min\left\{ \underset{(0,0)}{\overset{0+\infty}{c(0,0)+f_3(1,0)}},\ \underset{(1,0)}{\overset{5+0}{c(1,0)+f_3(0,0)}} \right\} = 5$$

$$f_2(0,2) = \min_{\substack{x_{21}\in\{0,1\}\\ x_{22}\in\{0,1\}\\ x_{21}\le 0\\ x_{22}\le 2}} \left\{ c(x_{21},x_{22}) + f_3(0-x_{21},\, 2-x_{22}) \right\} = \min\left\{ \underset{(0,0)}{\overset{0+\infty}{c(0,0)+f_3(0,2)}},\ \underset{(0,1)}{\overset{7+\infty}{c(0,1)+f_3(0,1)}} \right\} = +\infty$$

$$f_2(0,1) = \min_{\substack{x_{21}\in\{0,1\}\\ x_{22}\in\{0,1\}\\ x_{21}\le 0\\ x_{22}\le 1}} \left\{ c(x_{21},x_{22}) + f_3(0-x_{21},\, 1-x_{22}) \right\} = \min\left\{ \underset{(0,0)}{\overset{0+\infty}{c(0,0)+f_3(0,1)}},\ \underset{(0,1)}{\overset{7+0}{c(0,1)+f_3(0,0)}} \right\} = 7$$

$$f_2(0,0) = \min_{\substack{x_{21}\in\{0,1\}\\ x_{22}\in\{0,1\}\\ x_{21}\le 0\\ x_{22}\le 0}} \left\{ c(x_{21},x_{22}) + f_3(0-x_{21},\, 0-x_{22}) \right\} = \min\left\{ \underset{(0,0)}{\overset{0+0}{c(0,0)+f_3(0,0)}} \right\} = 0$$

Stage 1:

$$f_1(1,2) = \min_{\substack{x_{11}\in\{0,1\}\\ x_{12}\in\{0,1\}\\ x_{11}\le 1\\ x_{12}\le 2}} \left\{ c(x_{11},x_{12}) + f_2(1-x_{11},\, 2-x_{12}) \right\} = \min\left\{ \begin{array}{ll} \underset{(0,0)}{\overset{0+\infty}{c(0,0)+f_2(1,2)}}, & \underset{(1,0)}{\overset{5+\infty}{c(1,0)+f_2(0,2)}} \\ \underset{(0,1)}{\overset{7+14}{c(0,1)+f_2(1,1)}}, & \underset{(1,1)}{\overset{14+7}{c(1,1)+f_2(0,1)}} \end{array} \right\} = 21$$

Optimal value: $f_1(1,2) = 21$ $\Rightarrow$ Minimum total cost of building 1000 oil capacity + 2000 gas capacity = $21 million

Optimal solution: $(x_{11}, x_{12}) = (1,1)$ $\Rightarrow$ At location 1, build 1000 oil capacity
1000 gas capacity

$(x_{21}, x_{22}) = (0,1)$ $\Rightarrow$ At location 2, build 1000 gas capacity